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LETTER TO THE EDITOR

Relativistic effects selecting the superfluid phase of neutron star matter

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Abstract. The BCS free energy for $^3P_2$ paired neutron matter is derived taking account of relativistic effects. It is found that the values taken by the Ginzburg–Landau parameters are always in the region of the phase diagram corresponding to a unitary phase.

Densities in the cores of neutron stars are thought to be sufficient to produce $^3P_2$ paired neutron superfluid (Hoffberg *et al* 1970, Tamagaki 1970). This anisotropic superfluid can have a significant influence on the observable properties of a neutron star, e.g. by affecting the rate of cooling by neutrino emission (Maxwell *et al* 1978), and by affecting the characteristic time for the transfer of angular momentum between the interior and the surface of the star through interactions of electrons with vortex cores (Sauls and Stein 1981). It is, therefore, important to know which of the possible $^3P_2$ paired superfluid phases is realised. For different values of the parameters in the Ginzburg–Landau free energy, unitary phases and two distinct non-unitary phases are possible (see figure 1). With the (non-relativistic) BCS values of the parameters, the unitary phase region of the phase diagram (region III) is selected.

Sauls and Serene (1978) have investigated the possibility that strong-coupling corrections might instead select one of the non-unitary phases, but have found that these corrections are too small, and in any case go in the wrong direction. There is, however, a much larger effect that has not so far been investigated, namely, the modification of the Ginzburg–Landau parameters in the BCS theory by relativistic corrections. At a Fermi energy of about 100 MeV, we should have $(p_F/m)^2 \approx 0.2$ for neutrons, and 20% corrections to the Ginzburg–Landau parameters are to be expected. Corrections of this size, if they were to go in the right direction, could be adequate to move the system into the nearby region II of figure 1, where the order parameter is non-unitary. The purpose of this letter is to present a calculation of the BCS free energy for $^3P_2$ paired neutron matter, taking account of relativistic effects.

The relativistic gap equations may be derived from the Dyson equation for the proper self-energy of a neutron, using a method developed for non-relativistic superfluids by Nambu (1960). In this method, it is necessary to introduce an effective Lagrangian term

$$\mathscr{L}_\Delta = \bar{\psi}_c \Delta \psi \tag{1}$$

where ψ is the Dirac field of the neutron, ψ_c is the charge conjugate field, and the gap matrix Δ is a 4×4 matrix in Dirac spinor indices. The gap matrix Δ is then

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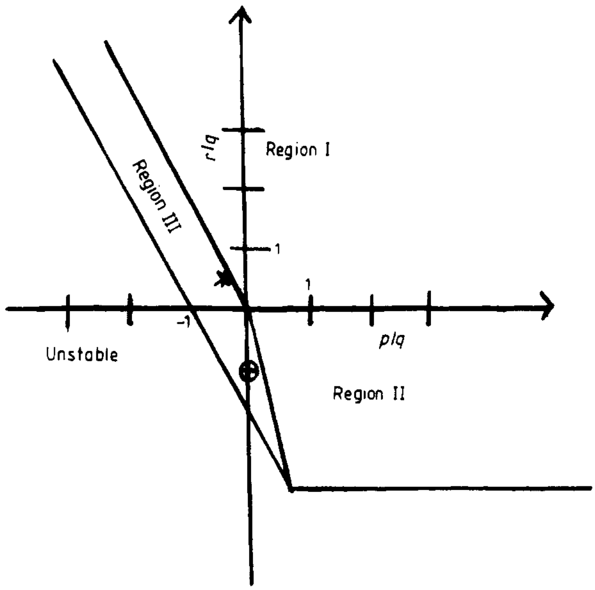

Figure 1. Phase diagram for the $^3P_2$ neutron superfluid. Region III corresponds to unitary phases, and regions I and III correspond to distinct non-unitary phases. The parameters p, q, r are as in equations (34)–(36). The non-relativistic limit is indicated by ○, and the ultra-relativistic limit is indicated by $*$.

calculated self-consistently from the Dyson equation. For the case of $^3P_2$ paired neutron superfluid the most general form of the gap matrix consistent with Fermi statistics is

$$\Delta(\boldsymbol{n}) = \Delta_{ij}^{(1)}T_{ij} + \Delta_{ij}^{(2)}S_{ij} + \Delta_{ij}^{(3)}Y_{ij} + \Delta_{ij}^{(4)}\tilde{S}_{ij} + \Delta_{ij}^{(5)}\tilde{Y}_{ij} + \Delta_{ij}^{(6)}X_{ij} \tag{2}$$

where

$$\boldsymbol{n} = \hat{\boldsymbol{k}} \tag{3}$$

for neutrons of momentum $\boldsymbol{k}$ and $-\boldsymbol{k}$ pairing, and the covariants S, $\tilde{S}$, T, X, Y, $\tilde{Y}$, constructed to have definite values of L, are defined to be

$$T_{ij} = n_i n_j - \tfrac{1}{3}\delta_{ij} \tag{4}$$

$$S_{ij} = \tfrac{1}{2}(n_i\gamma^0\gamma^j + n_j\gamma^0\gamma^i) - \tfrac{1}{3}\boldsymbol{n}\cdot\gamma^0\boldsymbol{\gamma}\,\delta_{ij} \tag{5}$$

$$\tilde{S}_{ij} = \tfrac{1}{2}(n_i\gamma^j + n_j\gamma^i) - \tfrac{1}{3}\boldsymbol{n}\cdot\boldsymbol{\gamma}\,\delta_{ij} \tag{6}$$

$$Y_{ij} = [n_i n_j n_k - \tfrac{1}{5}(n_i\delta_{jk} + n_j\delta_{ik} + n_k\delta_{ij})]\gamma^0\gamma^k \tag{7}$$

$$\tilde{Y}_{ij} = [n_i n_j n_k - \tfrac{1}{5}(n_i\delta_{jk} + n_j\delta_{ik} + n_k\delta_{ij})]\gamma^k \tag{8}$$

$$X_{ij} = \tfrac{1}{2}\mathrm{i}[(\boldsymbol{n}\times\boldsymbol{\gamma}\gamma_5)_i n_j + (\boldsymbol{n}\times\boldsymbol{\gamma}\gamma_5)_j n_i]. \tag{9}$$

The matrices $\Delta_{ij}^{(\rho)}$, $\rho = 1, \ldots, 6$, are symmetric and traceless because they describe $J = 2$ pairing.

The derivation of relativistic gap equations for $J = 0$ and $J = 1$ pairing has been discussed in a number of recent papers (Barrois 1979, Bailin and Love 1981a, b, 1982a, b) and we omit the (lengthy) details of the present $J = 2$ calculation. The gap equations for the individual matrices $\Delta_{ij}^{(\rho)}$ are highly model dependent, involving the detailed form of pairing force assumed. However, the gap equations for the possible order parameters may be expressed in terms of the helicity amplitudes for neutron–neutron scattering, in a form which is independent of the particular pairing force. In the first instance, the gap equations arise in terms of the coupled order parameters $d^{(\theta)}$, $\theta = 1, 2$, where

$$d^{(1)} = p_F\Delta^{(1)} + \mu(\Delta^{(2)} + \tfrac{3}{5}\Delta^{(3)}) - m(\Delta^{(4)} + \tfrac{3}{5}\Delta^{(5)}) \tag{10}$$

$$d^{(2)} = -m(\Delta^{(2)} - \tfrac{2}{5}\Delta^{(3)}) + \mu(\Delta^{(4)} - \tfrac{2}{5}\Delta^{(5)}) + p_F\Delta^{(6)}. \tag{11}$$

Here, $d^{(\theta)}$ and $\Delta^{(\rho)}$ are 3×3 matrices, and

$$\mu = (p_F^2 + m^2)^{1/2}. \tag{12}$$

The gap equations in the Ginzburg–Landau region for these coupled order parameters are

$$d^{(\theta)} = F^{\theta\phi}(A\, d^{(\phi)} + BD^{(\phi)}) \tag{13}$$

where

$$A = \tfrac{1}{6}(\mathrm{d}n/\mathrm{d}\varepsilon)\ln(\mathscr{S}\beta\varepsilon_0) \tag{14}$$

$$B = -\tfrac{7}{48}(\mathrm{d}n/\mathrm{d}\varepsilon)(\pi k_B T_c)^{-2}\mathscr{S}(3) \tag{15}$$

$$\mathrm{d}n/\mathrm{d}\varepsilon = \mu p_F/\pi^2 \tag{16}$$

and $F^{\theta\phi}$ is the matrix of helicity amplitudes

$$F = \frac{24\pi^2}{\mu p_F}\begin{pmatrix} f_{11}^2 & -\sqrt{\tfrac{3}{2}}f_{12}^2 \\ -\sqrt{\tfrac{2}{3}}f_{12}^2 & f_{22}^2 \end{pmatrix}. \tag{17}$$

The helicity amplitudes f_{11}^J etc are defined as in Goldberger *et al* (1960). Also, $D^{(\theta)}$, $\theta = 1, 2$, are 3×3 matrices cubic in $d^{(\theta)}$, defined by

$$\begin{aligned}63\mu^2 D^{(1)} = {} & -7\,\mathrm{Tr}[(d^{(2)})^2]d^{(1)*} + 14\,\mathrm{Tr}(d^{(2)*}d^{(2)})d^{(1)} + 2\,\mathrm{Tr}[(d^{(1)})^2]d^{(1)*} \\ & + 4\,\mathrm{Tr}(d^{(2)}d^{(1)*})d^{(2)} - 4\,\mathrm{Tr}(d^{(1)}d^{(2)})d^{(2)*} - 4\,\mathrm{Tr}(d^{(2)*}d^{(1)})d^{(2)} \\ & + 4\,\mathrm{Tr}(d^{(1)*}d^{(1)})d^{(1)} - 20(d^{(2)})^2 d^{(1)*} + 20 d^{(2)*}d^{(2)}d^{(1)} \\ & + 20 d^{(2)}d^{(2)*}d^{(1)} + 8 d^{(2)}d^{(1)*}d^{(2)} + 8 d^{(1)}d^{(1)*}d^{(1)} \\ & + 16 d^{(1)*}(d^{(1)})^2 + 16 d^{(2)*}d^{(1)}d^{(2)}\end{aligned} \tag{18}$$

$$\begin{aligned}189\mu^2 D^{(2)} = {} & 62\,\mathrm{Tr}(d^{(2)*}d^{(2)})d^{(2)} - 23\,\mathrm{Tr}[(d^{(2)})^2]d^{(2)*} + 28\,\mathrm{Tr}(d^{(1)*}d^{(1)})d^{(2)} \\ & - 8\,\mathrm{Tr}(d^{(2)}d^{(1)*})d^{(1)} - 8\,\mathrm{Tr}(d^{(1)}d^{(2)})d^{(1)*} + 8\,\mathrm{Tr}(d^{(1)}d^{(2)*})d^{(1)} \\ & - 14\,\mathrm{Tr}[(d^{(1)})^2]d^{(2)*} + 32 d^{(2)*}(d^{(2)})^2 - 2 d^{(2)}d^{(2)*}d^{(2)} + 40 d^{(1)*}d^{(1)}d^{(2)} \\ & + 40 d^{(1)}d^{(1)*}d^{(2)} - 32 d^{(1)*}d^{(2)}d^{(1)} - 40(d^{(1)})^2 d^{(2)*} + 16 d^{(1)}d^{(2)*}d^{(1)}.\end{aligned} \tag{19}$$

The gap equations may be diagonalised using the matrix

$$S^{-1} = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \tfrac{1}{2}[1 + z^2 + (1+z^2)^{1/2}]^{-1/2}\begin{pmatrix} 1 + (1+z^2)^{1/2} & -\sqrt{\tfrac{3}{2}}z \\ \sqrt{\tfrac{2}{3}}z & 1 + (1+z^2)^{1/2} \end{pmatrix} \tag{20}$$

where

$$z = 2f_{12}^2/(f_{22}^2 - f_{11}^2). \tag{21}$$

Thus

$$SFS^{-1} = \begin{pmatrix} \lambda^{(1)} & 0 \\ 0 & \lambda^{(2)} \end{pmatrix} \tag{22}$$

with

$$\lambda^{(1)} = (12\pi^2/\mu p_F)[f_{11}^2 + f_{22}^2 + (f_{11}^2 - f_{22}^2)(1+z^2)^{1/2}] \tag{23}$$

$$\lambda^{(2)} = (12\pi^2/\pi p_F)[f_{11}^2 + f_{22}^2 - (f_{11}^2 - f_{22}^2)(1+z^2)^{1/2}]. \tag{24}$$

We write for the order parameters diagonalising the gap equations

$$e^{(\theta)} = S^{\theta\phi} d^{(\phi)} \tag{25}$$

and we also write

$$E^{(\theta)} = S^{\theta\phi} D^{(\phi)}. \tag{26}$$

The diagonalised equations may be written as

$$\tfrac{1}{6}(\mathrm{d}n/\mathrm{d}\varepsilon) t^{(\theta)} e^{(\theta)} = BE^{(\theta)}, \qquad \theta = 1, 2, \tag{27}$$

where

$$t^{(\theta)} = (T - T_c^{(\theta)})/T_c^{(\theta)} \tag{28}$$

$$\ln(\mathscr{S}\beta_c^{(\theta)}\varepsilon_0) = (\tfrac{1}{6}\lambda^{(\theta)}\,\mathrm{d}n/\mathrm{d}\varepsilon)^{-1}. \tag{29}$$

The order parameter $e^{(\theta)}$ with the higher critical temperature is the one which orders at the phase transition. Then, in (27), in the gap equation for the relevant order parameter, the other order parameter should be set to zero. It is just as easy to handle both possibilities, $e^{(1)}$ orders or $e^{(2)}$ ordering in a single formula, so we do this and return later to the question of which of $e^{(1)}$ and $e^{(2)}$ orders in reality. Then we obtain the gap equation for the order parameter $e^{(\theta)}$

$$\tfrac{1}{6}(\mathrm{d}n/\mathrm{d}\varepsilon) te = (Bk/63\mu^2)[6(x^4 + x^2y^2 - 2y^4)\,\mathrm{Tr}(e^2)e^* + 6(2x^4 + 14x^2y^2 + 5y^4)\,\mathrm{Tr}(e^*e)e - 9(8x^2y^2 - y^4)(e^2e^* + e^*e^2)] \tag{30}$$

where

$$k^{(\theta)} = 1, \tfrac{2}{3} \qquad \text{for } \theta = 1, 2 \tag{31}$$

and we have suppressed the index θ on e, t, T_c, k, x, y.

In arriving at (30), we have used the identity for traceless 3×3 matrices e,

$$2ee^*e + 2e^2e^* + 2e^*e^2 = \mathrm{Tr}(e^2)e^* + 2\mathrm{Tr}(e^*e)e \tag{32}$$

to eliminate ee^*e. The Ginzburg–Landau free energy corresponding to the gap equation (30) is

$$\mathscr{F} \propto \tfrac{1}{6}(\mathrm{d}n/\mathrm{d}\varepsilon)t\,\mathrm{Tr}(e^*e) - (Bk/63\mu^2)\{p|\mathrm{Tr}(e^2)|^2 + q[\mathrm{Tr}(e^*e)]^2 + r\,\mathrm{Tr}(e^{*2}e^2)\} \tag{33}$$

where the constant of proportionality, which is inessential here, is not determined by the gap equation. Again, the index θ has been suppressed, and

$$p = 3(x^4 + x^2y^2 - 2y^4) \tag{34}$$

$$q = 3(2x^4 + 14x^2y^2 + 5y^4) \tag{35}$$

$$r = -9(8x^2y^2 - y^4). \tag{36}$$

The quantities x_θ and y_θ are as in (20).

To identify the order parameter for the realistic case of neutron star matter we consider the non-relativistic limit in which $z \to 2\sqrt{6}$. Then, using (25), (20), (10) and (11), we find that

$$e^{(1)} \to \sqrt{\tfrac{3}{5}}m(\Delta^{(3)} - \Delta^{(5)}) \tag{37}$$

$$e^{(2)} \to \sqrt{\tfrac{5}{3}}m(\Delta^{(4)} - \Delta^{(2)}). \tag{38}$$

Now, we see from (2) that in the non-relativistic limit $e^{(1)}$ and $e^{(2)}$ are respectively pure $L = 3$ and pure $L = 1$ order parameters. Thus, the realistic $^3P_2$ pairing is described by the order parameter $e^{(2)}$. Accordingly we shall now restrict attention to the Ginzburg–Landau free energy for the order parameter $e^{(2)}$. For this case, the non-relativistic limit of (34), (35) and (36) gives

$$p = 0, \qquad r = -q, \tag{39}$$

in agreement with Sauls and Serene (1978), and the system is in region III of figure 1, corresponding to a unitary phase. In general, the criterion for region III is

$$4p + 2|p| + r < 0. \tag{40}$$

It is easy to check that this criterion is always satisfied by the p, q and r of (34), (35) and (36), for the physically allowed values of z

$$0 \leqslant z \leqslant 2\sqrt{6}. \tag{41}$$

Thus, even after taking account of relativistic effects, the system is always in a unitary phase. This is despite a striking variation in the Ginzburg–Landau parameters in going from the non-relativistic limit, of (39), to the ultra-relativistic limit ($z \to 0$), where

$$r:q:p = 3:5:-2. \tag{42}$$

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